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ANALYTICAL STUDY OF HEAT TRANSFER TO LIQUID METALS FLOWING ALONG A ROW OF SPHERES*

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Abstract—On the postulation of inviscid potential flow, theoretical analyses have been performed for heat transfer to liquid metals flowing along a row of equally-spaced spheres, the pitch-to-diameter ratio of which ranges from unity (touching spheres) to infinity (a single sphere in the fluid stream). The following explicit expressions for the average Nusselt number were obtained:

$$\text{For uniform surface temperature: } \overline{Nu} = \left(\sqrt{\frac{2}{\pi}}\right) Pe^{\frac{1}{2}} (\phi'_{\pi}/R^3)^{\frac{1}{2}}.$$

$$\text{For uniform surface heat flux: } \overline{Nu}_D = 1{\cdot}0035\, Pe^{\frac{1}{2}} (\phi'_{\pi}/R^3)^{\frac{1}{2}}.$$

It was revealed that the surface velocity potential factor, ϕ'_{π}/R^3, which appears in the above equations is solely a function of the pitch-to-diameter ratio. A theoretical method has been developed for evaluating the numerical values of ϕ'_{π}/R^3 utilizing the vector potential recently obtained by Michael [9]. In connection with this, an equation describing the velocity potential along a row of spheres has been obtained. Numerical values of ϕ'_{π}/R^3 were determined by means of an IBM digital computer. It was found to vary from 2·0 for flow past a single sphere to $\approx$ 1·5639 for flow along a row of touching spheres.

NOMENCLATURE

A,	vector potential;
A_k^n,	series expansion coefficients in Legendre polynomials as defined by equation (16);
C,	heat capacity [Btu/lb degF];
C_n,	expansion coefficients in the stream function given by equation (5);
D,	diameter of a sphere [ft];
K,	thermal conductivity [Btu/ft s degF];
K',	as defined by equation (40a);
N,	an integer describing the upper limit of the coefficient, C_n;
Nu_L,	local Nusselt number, hD/K [dimensionless];
$\overline{Nu}$,	average Nusselt number for constant surface temperature case, $\bar{h}D/K$ [dimensionless];
$\overline{Nu}_D$,	average Nusselt number for constant surface heat flux, $\bar{h}_D D/K$ [dimensionless];
$\overline{Nu}_t$,	average Nusselt number based upon h_t, $h_t D/K$ [dimensionless];
P,	pitch [ft];
P_{2n+1}^1,	associated Legendre polynomials;
Pe,	Péclet number, $\rho CVD/K$ [dimensionless];
R,	radius of a sphere [ft];
R_j,	$= (1 + 4\lambda^2 j^2 - 4\lambda j \cos\theta)^{\frac{1}{2}}$;
T,	temperature [degF];
T_i,	uniform approaching temperature [degF];
T_s,	temperature excess on the surface of spheres [degF];
T',	temperature excess [degF];
V,	uniform approaching velocity [ft/s];
V,	velocity vector;
$h(\theta)$,	local heat-transfer coefficient in the spherical coordinates [Btu/ft^2 s degF];
$h(\theta')$,	local heat-transfer coefficients in the ϕ'–ψ coordinates [Btu/ft^3 s degF];
$\bar{h}$,	average heat-transfer coefficients for

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constant surface temperature case [Btu/ft^2 s degF];

$h_D(\theta)$, local heat-transfer coefficients for constant surface heat flux case [Btu/ft^2 s degF];

$\bar{h}_D(\theta)$, average heat-transfer coefficients, $= (1/(4\pi R^2) \iint_s h_D \, ds$ where s is the surface area of the sphere) [Btu/ft^2 s degF];

h_t, average heat-transfer coefficients based on $\bar{\theta}_0$, $= q''/\bar{\theta}_0$ [Btu/ft^2 s degF];

j, k, n, integers;

$q'(\phi')$, surface heat flux for constant surface temperature case in the ϕ'–ψ coordinates [Btu/ft^3 s];

q'', surface heat flux in the spherical coordinates [Btu/ft^2 s];

$q''(\phi')$, surface heat flux in the ϕ'–ψ coordinates [Btu/ft^3 s];

r, r_j, radial distance variables as defined in Fig. 1 [ft];

v_r, v_θ, velocity components in the r and θ direction, respectively.

Greek symbols

α, $\lambda R = P/2$ [ft];

β, an angle in the spherical coordinates [degree or rad];

Γ_k^n, as defined by equation (19);

η, a parameter;

η_1, $1 + 2\lambda j$;

η_2, $1 - 2\lambda j$;

θ, angle measured from the front stagnation point of a sphere on the x–y plane [degree or rad];

θ_0, local surface temperature excess for constant surface heat flux case [degF];

$\bar{\theta}$, average surface temperature excess $= (1/4\pi R^2) \iint_s \theta_0 \, ds$ where s is the surface area of the sphere [degF];

θ_j, angles as defined in Fig. 1 [degree or rad];

λ, P/D, pitch to diameter ratio [dimensionless];

ξ, as defined by equation (40d);

π, $= 3{\cdot}1416\ldots$;

ρ, density of fluid [lb/ft^3];

τ, a parameter;

Φ, hydrodynamic potential function [ft^2/s];

ϕ, Φ/V [ft];

ϕ_π, ϕ at the rear stagnation point of a sphere [ft];

ϕ', as defined by equation (9) [ft^3];

ϕ'_π, ϕ' at the rear stagnation point of a sphere [ft^3];

Ψ, stream function [ft^3/s];

ψ, Ψ/V [ft^2];

ω, as defined by equation (40b).

INTRODUCTION

THEORETICAL prediction of the characteristics of energy transport between orderly arrayed spheres and liquid metals flowing past them is important because of its possible application in the design of certain types of nuclear reactors such as the ordered-bed reactor, whose fuel consists of spherical beads. In general, strictly analytical treatment of such a three-dimensional heat-transfer problem involves a great deal of difficulty due to the complex flow patterns associated with the configurations of the spheres. As a step toward a better understanding of such intricate problems, the present paper considers the heat-transfer problem for one of the fundamental flow geometries, i.e. for flow along a row of equally spaced spheres where the spacing ranges from zero to infinity. For this geometry, both velocity and temperature profiles become axially symmetric, and the problem is simplified to some extent. As shown in previous papers [7, 8], the assumption of inviscid flow is essentially valid for analyzing the heat transfer to liquid metals flowing past submerged bodies, except at very high flow rates. This is mainly due to the fact that the high thermal conductivities of liquid metals overshadow eddy transport of heat even in the wake region. As pointed out previously, it has been experimentally observed [1, 5] that, for cross-flow of liquid metals such as mercury or liquid sodium past a cylindrical rod, the local heat-transfer coefficient gradually decreases toward a minimum as the rear stagnation point is approached. For ordinary fluids, this may not be so because eddy transport of heat in the wake region may become significant enough to cause a substantial increase in the local heat-transfer coefficient at the rear part of the cylinder. For flow of liquid metals past spheres, a situation similar to that for flow past cylinders can be expected to prevail. Since the assumption of inviscid potential flow could lead to theoretical equations which agree well with experimental results as shown previously [7] the same assumption will be made in this analysis. Other conventional assumptions imposed in the previous analyses also apply to the present work.

The stream function for potential flow along a row of spheres has been obtained by Howland [6] utilizing certain types of periodic functions and recently by Michael [9] who made use of an electromagnetic analogy and obtained the solution in terms of a vector potential **A** (the velocity **V** = −curl **A**). In the present heat-transfer analysis, the latter velocity field was utilized because it has the advantage of relatively rapid convergence even for the spheres with zero spacing. The expressions for both local and average Nusselt numbers have been derived for the boundary conditions of uniform surface temperature and uniform surface heat flux, the two most commonly encountered thermal boundary conditions. Analogous to the case of flow across a cluster of rods [7] these expressions contain a surface velocity potential factor, ϕ'_π/R^3, a quantity which is intimately related to the velocity potential difference between the front and rear stagnation points of a sphere located inside the row. An explicit equation has been derived for this quantity and numerical values were obtained with the aid of a high-speed digital computer. The computational results are presented as a function of the pitch-to-diameter ratio, P/D, and the effect of sphere spacing on the local and average Nusselt numbers is examined and discussed.

THEORETICAL ANALYSES

Consideration is given to the potential flow of fluid along a row of spheres as illustrated in Fig. 1. It is apparent that both the velocity and temperature fields are rotationally symmetric about the x–x' axis so that only variations in the radial and azimuthal directions are significant. Furthermore, by virtue of the periodic symmetry in the axial direction, it suffices to consider the velocity fields around the sphere within a half of the shaded area ($0 \leqslant \theta \leqslant \pi/2$) shown in Fig. 1. For temperature fields, however, the entire shaded area ($0 \leqslant \theta \leqslant \pi$) must be taken into consideration. Because of the spherical boundary, it is advantageous to adopt the spherical coordinates. Under the aforementioned assumptions, the energy equation can be written as:

$$v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r}\frac{\partial T}{\partial \theta} = \frac{K}{C\rho}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial T}{\partial \theta}\right)\right]. \quad (1)$$

where the velocity components, v_r and v_θ can, in principle, be obtained by solving the Laplace equation:

$$\operatorname{div}\operatorname{grad}\Phi = 0 \quad (2)$$

for this particular flow geometry and then making use of the relationships:

$$v_r = \frac{\partial \Phi}{\partial r} = \frac{1}{r^2 \sin\theta}\frac{\partial \Psi}{\partial \theta} \quad (3)$$

$$v_\theta = \frac{1}{r}\frac{\partial \Phi}{\partial \theta} = -\frac{1}{r\sin\theta}\frac{\partial \Psi}{\partial r}. \quad (4)$$

As mentioned earlier, the velocity field for potential flow along a row of spheres has been obtained by Michael [9] in terms of a vector potential. His solution can be visualized more

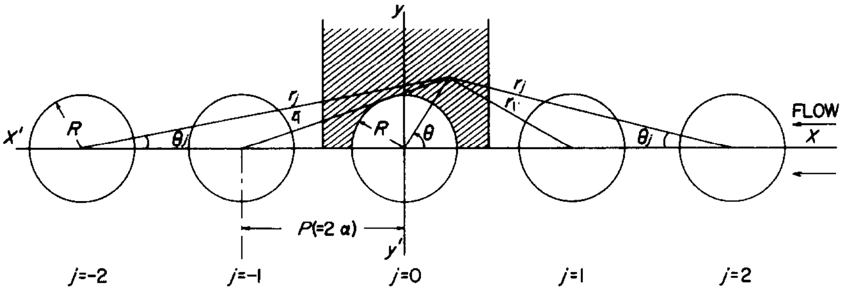

FIG. 1. Geometry for flow along a row of spheres.

easily in terms of the stream function, ψ, which, after a slight modification, can be expressed as

$$\psi = -\tfrac{1}{2}r^2\sin^2\theta + \sum_{n=0}^{N} C_n \sum_{j=-\infty}^{\infty} \times \frac{R^{2n+3}\, r\sin\theta\, P_{2n+1}^{1}(\cos\theta_j)}{r_j^{2n+2}}. \quad (5)$$

In the above equation, P_{2n+1}^{1} is the associated Legendre polynomials and r_j and θ_j are defined in Fig. 1. The coefficients for the infinite series, C_n, were evaluated by Michael for several values of $\lambda = (P/D)$. Additional results for other values of λ are tabulated in Table 1.* Owing to the complex expression of the velocity fields, the direct mathematical solution of equations (1–5) is arduous, if not impossible, to obtain. A considerable simplification can be achieved, however, if the independent variables are transformed from r and θ to Ψ and Φ. This is the so-called "Boussinesq transformation", and it transforms, in effect, the three dimensional energy equation of the form:

$$\mathbf{V}\cdot\nabla T = (K/C\rho)\,\mathrm{div\,grad}\,T \quad (6)$$

into the equation of the form:

$$\frac{\partial T}{\partial \Phi} = \left(\frac{K}{C\rho}\right)\left[\left(\frac{\nabla\Psi}{\nabla\Phi}\right)^2 \frac{\partial^2 T}{\partial \Psi^2} + \frac{\nabla^2\Psi}{(\nabla\Phi)^2}\frac{\partial T}{\partial\Psi} + \frac{\partial^2 T}{\partial\Phi^2}\right]. \quad (7)$$

Thus, applying the Boussinesq transformation, equation (1) can finally be transformed to [2]:

$$V\frac{\partial T}{\partial\phi} = \frac{K}{C\rho}\left[r^2\sin^2\theta\frac{\partial^2 T}{\partial\psi^2}\right]. \quad (8)$$

Geometrically, the transformation causes the circular boundary to be mapped into a line segment and gives rise to a flow with constant unidirectional velocity, V. Defining a new independent variable, ϕ', by the relationship:

$$\phi' = \int r^2\sin^2\theta\,\mathrm{d}\phi \quad (9)$$

and changing the temperature variable by letting $T' = T - T_i$, equation (8) can further be simplified to the following equation.

$$V\frac{\partial T'}{\partial\phi'} = \frac{K}{C\rho}\frac{\partial^2 T'}{\partial\psi^2}. \quad (10)$$

The two consecutive changes of variables thus transform the energy equation written in terms of the r–θ coordinates to that in terms of the ϕ'–ψ coordinates. Since mathematical solutions to equation (10) for various boundary conditions are available [3] attempts can now be made to seek the desired temperature solutions on the ϕ'–ψ plane and then transform them back to those on the r–θ plane. Before proceeding to obtain the temperature solutions, however, the analytical evaluation of the surface velocity potential factor, ϕ'_π/R^3 will first be discussed. Determination of the numerical values

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Table 1. *The constants*, C_n, *in equation* (5)

$\lambda = 1.1$		$\lambda = 1.2$		$\lambda = 1.3$	
n	C_n	n	C_n	n	C_n
0	0.41101	0	0.42740	0	0.44055
1	-0.014473	1	-1.0356×10^{-2}	1	-7.3837×10^{-3}
2	-0.0024499	2	-1.6094×10^{-3}	2	-1.01711×10^{-3}
3	-3.5703×10^{-4}	3	-2.4157×10^{-4}	3	-1.3988×10^{-4}
4	-2.8275×10^{-5}	4	-3.2708×10^{-5}	4	-1.8654×10^{-5}
5	7.9719×10^{-6}	5	-3.3772×10^{-6}	5	-2.3461×10^{-6}
6	5.7585×10^{-6}	6	-1.4708×10^{-8}	6	-2.6308×10^{-7}
7	2.3812×10^{-6}	7	1.3864×10^{-7}		
8	8.2472×10^{-7}	8	5.8769×10^{-8}		
9	2.5823×10^{-7}	9	1.8688×10^{-8}		
10	7.4375×10^{-8}				
11	1.9423×10^{-8}				
12	4.3122×10^{-9}				
13	6.1529×10^{-10}				
14	-9.8216×10^{-11}				

$\lambda = 1.4$		$\lambda = 1.6$		$\lambda = 1.8$	
n	C_n	n	C_n	n	C_n
0	0.451006	0	0.46592	0	0.47553
1	-5.3067×10^{-3}	1	-2.8535×10^{-3}	1	-1.6245×10^{-3}
2	-6.4245×10^{-4}	2	-2.6859×10^{-4}	2	-1.2144×10^{-4}
3	-7.8601×10^{-5}	3	-2.5748×10^{-5}	3	-9.2649×10^{-6}
4	-9.5174×10^{-6}	4	-2.4702×10^{-6}	4	-7.092×10^{-7}
5	-1.1291×10^{-6}	5	-2.3608×10^{-7}	5	-5.4259×10^{-8}
6	-1.2971×10^{-7}	6	-2.2428×10^{-8}	6	-4.1445×10^{-9}

$\lambda = 2.2$		$\lambda = 2.4$		$\lambda = 2.6$	
n	C_n	n	C_n	n	C_n
0	0.48628	0	0.48936	0	0.49160
1	-6.1092×10^{-4}	1	-3.9809×10^{-4}	1	-2.6807×10^{-4}
2	-3.0659×10^{-5}	2	-1.6794×10^{-5}	2	-9.6378×10^{-6}
3	-1.5718×10^{-6}	3	-7.2387×10^{-7}	3	-3.5406×10^{-7}
4	-8.0949×10^{-8}	4	-3.1346×10^{-8}	4	-1.3069×10^{-8}

$\lambda = 2.8$		$\lambda = 3.0$		$\lambda = 3.5$	
n	C_n	n	C_n	n	C_n
0	0.49325	0	0.49450	0	0.49652
1	-1.8571×10^{-4}	1	-1.3187×10^{-4}	1	-6.1264×10^{-5}
2	-5.7576×10^{-6}	2	-3.5617×10^{-6}	2	-1.2158×10^{-6}
3	-1.8241×10^{-7}	3	-9.8301×10^{-8}	3	-2.4655×10^{-8}
4	-5.8062×10^{-9}	4	-2.7260×10^{-9}	4	-5.0237×10^{-10}

$\lambda = 4.0$	
n	C_n
0	0.49766
1	-3.1496×10^{-5}
2	-4.7856×10^{-7}
3	-7.4303×10^{-9}
4	-1.1592×10^{-10}

of this factor is of vital importance in connection with the present heat-transfer analyses.

$$\phi'_\pi/R^3 = \frac{1}{R}\int_0^\pi \sin^2\theta\, \mathrm{d}\phi. \tag{11}$$

A. *Analytical calculation of the surface velocity potential factor, ϕ'_π/R^3*

For heat transfer in flow along a row of spheres, the theoretical Nusselt numbers contain, as will be shown later, a proportional factor, ϕ'_π/R^3, which will be called the surface velocity potential factor. This quantity arises as a consequence of the coordinates transformations and is related to the velocity potential on the surface of the spheres, ϕ, by means of the following equation:

It may be noted that equation (11) is obtained by substituting $r = R$ in equation (9), integrating between the limit $(0, \pi)$. and then dividing through by R^3. Analytical evaluation of the above integral apparently requires an explicit expression of the velocity potential, ϕ, which was obtained in this study by making use of equation (5). To derive the theoretical expression for ϕ, the functional relationships among the variables, r, θ, r_j and θ_j were first explored. From Fig. 1, it is evident that the following relationships hold:

$$r \sin\theta = r_j \sin\theta_j \qquad \text{for all } j \text{ values} \tag{12}$$

$$r_j \cos\theta_j + r\cos\theta = 2|j|\alpha \text{ for } j \geqslant 1 \tag{13}$$

$$r_j \cos\theta_j - r\cos\theta = 2|j|\alpha \text{ for } j \leqslant 0 \tag{14}$$

where $\alpha = \lambda R$. Combining the above three equations and solving for r_j in terms of r and θ, the following equation results:

$$r_j = \sqrt{(r^2 + 4\alpha^2 j^2 - 4\alpha j r\cos\theta)} \tag{15}$$

for all j values. For mathematical convenience, the cosine terms, $\cos\theta_j$, in the associated Legendre polynomials appearing in equation (5) were converted to sine terms by writing:

$$P^1_{2n+1}(\cos\theta_j) = \sin\theta_j \sum_{k=0}^{n} A^n_k \sin^{2k}\theta_j. \tag{16}$$

Evaluation of the coefficients, A^n_k, was carried out, for each n, by the binomial expansion of the Legendre polynomials. As the values of n and k increase, the coefficients, A^n_k, become considerably larger, requiring accordingly higher computational accuracies. The computed values, for instance, are: $A^0_0 = 1, A^1_0 = 6, A^1_1 = -7{\cdot}5, A^2_0 = 15, A^2_1 = -52{\cdot}5, A^2_2 = 39{\cdot}375, \ldots, A^{14}_0 \cong 4{\cdot}349724 \times 10^2, \ldots, A^{14}_{14} \cong 1{\cdot}6241348 \times 10^9$, etc.

By combining equations (4, 5, 16), and carrying out the required differentiations and integrations, the following theoretical expression for the velocity potential, ϕ, for potential flow along a row of spheres finally results:

$$\phi = -\int \frac{1}{\sin\theta}\frac{\partial\psi}{\partial r}\,\mathrm{d}\theta = r\cos\theta + \sum_{n=0}^{N} C_n R^{2n+3} \sum_{j=-\infty}^{\infty} \sum_{k=0}^{n} A^n_k \Bigg\{ 2(k+1)\, r^{2k+1} \int \frac{\sin^{2k+1}\theta\,\mathrm{d}\theta}{r_j^{2n+2k+3}} - (2n+2k+3)\, r^{2k+3} \int \frac{\sin^{2k+1}\theta\,\mathrm{d}\theta}{r_j^{2n+2k+5}}$$
$$+ 2\alpha j(2n+2k+3)\, r^{2k+2} \int \frac{\sin^{2k+1}\theta\cos\theta\,\mathrm{d}\theta}{r_j^{2n+2k+5}} \Bigg\} \tag{17}$$

where r_j is defined by equation (15). The integrals appearing in the above equation can be evaluated analytically with the aid of recurrence formulas. It is interesting to note that, for flow past a single sphere, for which $j = 0$, $C_0 = \frac{1}{2}$, $C_n = 0\ (n = 1, 2, 3, \ldots)$, and $A_0^0 = 1$, equation (17) simplifies to:

$$\phi = r\cos\theta\left(1 + \frac{R^3}{2r^3}\right) \tag{18}$$

which is the well known velocity potential equation for flow past a single sphere [10].

The purpose at present is to derive an analytical expression for the surface velocity potential factor, ϕ'_π/R^3. The following equation was ultimately obtained by combining equation (11) with equation (17) and performing the required integrations.

$$\begin{aligned}\phi'_\pi/R^3 &= \frac{1}{R}\int_0^\pi \sin^2\theta\,\mathrm{d}\phi = \frac{1}{R}\int_0^\pi \sin^2\theta\left(\frac{\partial\phi}{\partial\theta}\right)\mathrm{d}\theta = \frac{4}{3} - \sum_{n=0}^{N} C_n \sum_{j=-\infty}^{\infty}\sum_{k=0}^{n} A_k^n\Bigg\{2(k+1)\int_0^\pi \frac{\sin^{2k+3}\theta\,\mathrm{d}\theta}{R_j^{2n+2k+3}} \\ &\quad -(2n+2k+3)\int_0^\pi \frac{\sin^{2k+3}\theta\,\mathrm{d}\theta}{R_j^{2n+2k+5}} + 2\lambda j(2n+2k+3)\int_0^\pi \frac{\sin^{2k+3}\theta\cos\theta\,\mathrm{d}\theta}{R_j^{2n+2k+5}}\Bigg\} = \frac{4}{3} - \sum_{n=0}^{N} C_n \sum_{j=-\infty}^{\infty}\sum_{k=0}^{n} \\ &\quad A_k^n\Bigg\{2(k+1)\int_0^\pi \frac{\sin^{2k+1}\theta\,\mathrm{d}\theta}{R_j^{2n+2k+3}} - (2n+2k+3)\int_0^\pi \frac{\sin^{2k+3}\theta\,\mathrm{d}\theta}{R_j^{2n+2k+5}} \\ &\quad - \int_0^\pi \frac{\sin^{2k+3}\theta\,\mathrm{d}\theta}{R_j^{2n+2k+3}}\Bigg\} = \frac{4}{3} - \sum_{n=0}^{N} C_n \sum_{j=-\infty}^{\infty}\sum_{k=0}^{n} A_k^n\Gamma_k^n\end{aligned} \tag{19}$$

where $\lambda = P/D$, and $R_j = \sqrt{(1 + 4\lambda^2 j^2 - 4\lambda j\cos\theta)}$. In simplifying the above equation, the following equality, obtainable by use of recurrence formulas, has been used:

$$2\lambda j(2n+2k+3)\int_0^\pi \frac{\sin^{2k+3}\theta\cos\theta\,\mathrm{d}\theta}{R_j^{2n+2k+5}} = 2(k+1)\int_0^\pi \frac{\sin^{2k+1}\theta\,\mathrm{d}\theta}{R_j^{2n+2k+3}} - (2k+3)\int_0^\pi \frac{\sin^{2k+3}\theta\,\mathrm{d}\theta}{R_j^{2n+2k+3}}.$$

The expressions for Γ_k^n in equation (19) are, for example,

$$\begin{aligned}\Gamma_0^n &= 2\int_0^\pi \frac{\sin\theta\,\mathrm{d}\theta}{R_j^{2n+3}} - (2n+3)\int_0^\pi \frac{\sin^3\theta\,\mathrm{d}\theta}{R_j^{2n+5}} - \int_0^\pi \frac{\sin^3\theta\,\mathrm{d}\theta}{R_j^{2n+3}} = \frac{1}{\lambda j(2n+1)}\left(\frac{1}{\eta_2^{2n+1}} - \frac{1}{\eta_1^{2n+1}}\right) \\ &\quad - \frac{1}{2\lambda^2 j^2(2n+1)(2n-1)}\left[\left(\frac{1}{\eta_1^{2n-1}} + \frac{1}{\eta_2^{2n-1}}\right) + \frac{1}{2\lambda j(2n-3)}\left(\frac{1}{\eta_1^{2n-3}} - \frac{1}{\eta_2^{2n-3}}\right)\right] \\ &\quad - \frac{1}{2\lambda^2 j^2(2n+1)}\left[\left(\frac{1}{\eta_1^{2n+1}} + \frac{1}{\eta_2^{2n+1}}\right) + \frac{1}{2\lambda j(2n-1)}\left(\frac{1}{\eta_1^{2n-1}} - \frac{1}{\eta_2^{2n-1}}\right)\right]\end{aligned} \tag{20}$$

in which $\eta_1 = 1 + 2\lambda j$, $\eta_2 = 1 - 2\lambda j$, and

$$\Gamma_1^n = 4\int_0^\pi \frac{\sin^3\theta\,\mathrm{d}\theta}{R_j^{2n+5}} - (2n+5)\int_0^\pi \frac{\sin^5\theta\,\mathrm{d}\theta}{R_j^{2n+7}} - \int_0^\pi \frac{\sin^5\theta\,\mathrm{d}\theta}{R_j^{2n+5}}$$

$$= \frac{1}{\lambda^2 j^2 (2n+1)(2n+3)} \left\{ \frac{3}{2\lambda^2 j^2 (2n-1)(2n-3)} \left[\left(\frac{1}{\eta_1^{2n-3}} + \frac{1}{\eta_2^{2n-3}} \right) \right. \right.$$

$$\left. \left. + \frac{1}{2\lambda j(2n-5)} \left(\frac{1}{\eta_1^{2n-5}} - \frac{1}{\eta_2^{2n-5}} \right) \right] + \frac{1}{\lambda j(2n-1)} \left(\frac{1}{\eta_1^{2n-1}} - \frac{1}{\eta_2^{2n-1}} \right) \right\}$$

$$+ \frac{1}{\lambda^2 j^2 (2n+3)} \left\{ \frac{3}{2\lambda^2 j^2 (2n+1)(2n-1)} \left[\left(\frac{1}{\eta_1^{2n-1}} + \frac{1}{\eta_2^{2n-1}} \right) \right. \right.$$

$$\left. \left. + \frac{1}{2\lambda j(2n-3)} \left(\frac{1}{\eta_1^{2n-3}} - \frac{1}{\eta_2^{2n-3}} \right) \right] + \frac{1}{\lambda j(2n+1)} \left(\frac{1}{\eta_1^{2n+1}} - \frac{1}{\eta_2^{2n+1}} \right) \right\}$$

$$+ \frac{2}{\lambda^2 j^2 (2n+1)(2n+3)} \left[\left(\frac{1}{\eta_1^{2n+1}} + \frac{1}{\eta_2^{2n+1}} \right) + \frac{1}{2\lambda j(2n-1)} \left(\frac{1}{\eta_1^{2n-1}} - \frac{1}{\eta_2^{2n-1}} \right) \right] \text{ etc.} \quad (21)$$

Explicit expressions of Γ_k^n are thus seen to become increasingly complex as k becomes larger. On substituting $C_0 = \frac{1}{2}$, $j = 0$, $A_0^0 = 1$, and $R_j = 1$, equation (19) gives, for flow past a single sphere, $\phi'_\pi/R^3 = 2{\cdot}0$. For flow along a row of spheres with a pitch-to-diameter ratio of λ, numerical values of ϕ'_π/R^3 were computed using equation (19), with the aid of the computer. The pitch-to-diameter ratio, λ, was varied from unity (corresponding to a row of touching spheres) to infinity (corresponding to the flow past a single sphere), while the numerical constants, C_n's, were taken from Table 1 and [9]. To assure satisfactory convergence of the infinite series appearing in equation (19), a sufficient number of terms were included in the computation, for each preassigned value of λ. In general, the number of terms required in obtaining a converged solution increases as the value of λ is decreased. It was necessary, for instance, to evaluate several hundred of the integrals appearing in equation (19), in order to obtain the converged solution corresponding to λ of unity. the entire computational results are tabulated in Table 2. The usefulness of these numerical results will become self-evident in the next section.

Table 2. Calculated values of the surface velocity potential factor, ϕ'_π/R^3

$\lambda = P/D$	ϕ'_π/R^3
∞	2·0
15·0	1·99982
10·0	1·99939
5·0	1·99520
4·0	1·99065
3·5	1·98607
3·0	1·97798
2·8	1·97299
2·6	1·96638
2·4	1·95745
2·2	1·94512
2·0	1·92763
1·8	1·90212
1·6	1·86370
1·5	1·83723
1·4	1·80402
1·3	1·76221
1·2	1·70961
1·1	1·64405
1·0	1·56393

B. *Derivation of Nusselt numbers for flow along a row of spheres*

In the following heat-transfer analysis, it will be assumed that there is no interaction of thermal boundary layers for the adjacent spheres. It will also be postulated that a similarity exists between the angular dependence of the surface velocity potential variable, ϕ', as defined by equation (9), for flow along a row of spheres and that for flow past a single sphere. To elaborate on the latter assumption, it is convenient to express the velocity potential distribution on the surface of a sphere for the latter case by the equation

$$\phi = \frac{3R}{2}(1 - \cos\theta) = \frac{\phi_\pi}{2}(1 - \cos\theta), \tag{22}$$

where $\phi_\pi(=3R)$ is the velocity potential at the rear stagnation point ($\theta = \pi$) of the sphere. Combining equation (9) with (22) and then dividing through by R^3, one obtains:

$$\frac{\phi'}{R^3} = \frac{1}{R}\int_0^\phi \sin^2\theta \, d\phi = \frac{\phi_\pi}{6R}(\cos^3\theta - 3\cos\theta + 2) = \frac{\phi'_\pi}{4R^3}(\cos^3\theta - 3\cos\theta + 2) \tag{23}$$

in which $\phi'_\pi/R^3(= 2\phi_\pi/3R)$ is the surface velocity potential factor, i.e. ϕ'/R^3 evaluated at the rear stagnation point ($\theta = \pi$) of the sphere. Equation (23) thus relates, for flow past a single sphere, ϕ'/R^3 at any angle θ to that at the rear stagnation point, ϕ'_π/R^3. The assumption will now be made that equation (23) is also valid for flow along a row of spheres provided that ϕ'_π/R^3 in equation (23) is replaced by that for flow along the row of spheres being considered. An analogous assumption was made previously [7] in extending the heat-transfer analysis for flow past a single rod to that for flow across rod-bundles. In fact, the validity of this assumption has been justified by actually finding out the angular dependence of ϕ'/R^3 using equations (9) and (17). Preliminary but laborious calculations revealed that this assumption is reasonably valid for most of the λ values being considered. Based upon this and other conventional assumptions made in the previous study [7, 8] the temperature solutions corresponding to the boundary condition of uniform temperature or uniform heat flux on the surface of the spheres have been obtained respectively as follows:

1. *Uniform temperature on the surface of the spheres.* For the case in which a uniform temperature is maintained on the surface of the sphere, the distribution of the heat flux on the surface can be readily obtained by solving equation (10) for this specific boundary condition. It is given as [3]:

$$q'(\phi') = T_s\sqrt{(\rho CVK/\pi\phi')} \tag{24}$$

from which

$$h(\phi') = \sqrt{(\rho CVK/\pi\phi')}. \tag{25}$$

Inasmuch as the above heat-transfer coefficient, $h(\phi')$, is based upon a unit area on the ϕ'–ψ plane, it must be converted to that based on the spherical coordinates, $h(\theta)$. These two heat-transfer coefficients are related by the following equation:

$$h(\phi')\, d\phi' = R^2\, h(\theta) \sin\theta \, d\theta. \tag{26}$$

Accordingly, if equations (25) and (23) are incorporated into equation (26), one obtains, after simplifying

$$h(\theta) = \frac{h(\phi')}{R^2 \sin\theta}\frac{d\phi'}{d\theta} = \frac{3}{(\sqrt{2\pi})}(\phi'_\pi/R^3)^{\frac{1}{2}}\left(\frac{\rho CVK}{D}\right)^{\frac{1}{2}} \frac{\sin^2\theta}{\sqrt{(\cos^3\theta - 3\cos\theta + 2)}} \tag{27}$$

from which the expression for the local Nusselt number can be obtained as

$$Nu_L(\theta) = \frac{h(\theta)D}{K} = \frac{3}{(\sqrt{2\pi})}(\phi'_\pi/R^3)^{\frac{1}{2}}\,Pe^{\frac{1}{2}}\frac{\sin^2\theta}{\sqrt{(\cos^3\theta - 3\cos\theta + 2)}}. \tag{28}$$

As pointed out earlier, for flow past a single sphere, $\phi'_\pi/R^3 = 2{\cdot}0$, and hence, for this particular case, equation (28) reduces to

$$Nu_L(\theta) = \frac{3}{(\sqrt{\pi})}Pe^{\frac{1}{2}}\frac{\sin^2\theta}{\sqrt{(\cos^3\theta - 3\cos\theta + 2)}}. \tag{29}$$

To derive the Nusselt number averaged over the entire surface of the sphere, the average heat-transfer coefficient, $\bar{h}$, is first obtained.

$$\bar{h}(\theta) = \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} h(\theta)\sin\theta\,d\beta\,d\theta = \frac{3}{4}\left(\sqrt{\frac{2}{\pi}}\right)(\phi'_\pi/R^3)^{\frac{1}{2}}(\rho CVK/D)^{\frac{1}{2}}$$

$$\int_0^{\pi}\frac{\sin^3\theta\,d\theta}{\sqrt{(\cos^3\theta - 3\cos\theta + 2)}} = \left(\sqrt{\frac{2}{\pi}}\right)(\phi'_\pi/R^3)^{\frac{1}{2}}(\rho CVK/D)^{\frac{1}{2}}. \tag{30}$$

The expression for the average Nusselt number, $\overline{Nu}$, therefore becomes

$$\overline{Nu} = \frac{\bar{h}D}{K} = \left(\sqrt{\frac{2}{\pi}}\right)Pe^{\frac{1}{2}}(\phi'_\pi/R^3)^{\frac{1}{2}}. \tag{31}$$

By substituting $\phi'_\pi/R^3 = 2{\cdot}0$ into equation (31), one obtains the average Nusselt number for flow past a single sphere, i.e.

$$\overline{Nu} = \frac{2}{(\sqrt{\pi})}Pe^{\frac{1}{2}}*. \tag{32}$$

It is interesting to note that equation (32) agrees exactly with the result of Boussinesq's analysis for heat-transfer to flow past a single sphere (2). In his analysis, however, only the average heat-transfer coefficient corresponding to the uniform surface temperature case is given. For flow along a row of spheres with variable pitch-to-diameter ratio, the appropriate value of ϕ'_π/R^3 to be used in equation (31) can be found in Table 2. The significance of the analysis presented in the previous section and the usefulness of Table 2 thus become obvious.

2. *Uniform heat flux on the surface of the spheres.* Consideration is next given to the case where uniform heat flux, q'', is maintained on the surfaces of the spheres. Because the heat flux is area-dependent, it is necessary to transform q'' to that based upon the ϕ'–ψ coordinates, $q''(\phi')$. The following relation exists between the two types of heat fluxes:

$$q''R^2\sin\theta\,d\theta = q''(\phi')\,d\phi'. \tag{33}$$

Hence, from equations (33) and (23),

$$q''(\phi') = q''R^2\sin\theta\frac{d\theta}{d\phi'} = \frac{4R^2q''}{3\phi'_\pi\sin^2\theta}. \tag{34}$$

* The coefficients for the Nusselt numbers given in Table 1 of [8] are in error because the variable transformation shown by equation (9) of this paper was not made. For uniform surface temperature case, the coefficient should be that given by equation (32) in this paper.

It is necessary to change the sine term appearing in the above equation to that in terms of ϕ'. To do this, the cosine terms in equation (23) are converted to sine terms, yielding

$$\sin^6\theta + 3\sin^4\theta + 4\left\{\left(1 - \frac{2\phi'}{\phi'_\pi}\right)^2 - 1\right\} = 0. \tag{35}$$

Solving the above equation for $\sin^2\theta$, one obtains

$$\sin^2\theta = 2\cos\left[\frac{\cos^{-1}\left\{1 - 2\left(1 - \frac{2\phi'}{\phi'_\pi}\right)^2\right\}}{3}\right] - 1. \tag{36}$$

Substituting this into equation (34) gives

$$q''(\phi') = \frac{4R^2}{3\phi'_\pi}\frac{q''}{2\cos\left[\frac{\cos^{-1}\left\{1 - 2\left(1 - \frac{2\phi'}{\phi'_\pi}\right)^2\right\}}{3}\right] - 1.}. \tag{37}$$

The solution of equation (10) corresponding to the surface heat flux distribution given by equation (37) can be written as [3]:

$$\theta_0(\phi') = (\pi\rho CVK)^{-\frac{1}{2}}\int_0^{\phi'} q''(\phi' - \tau)\frac{d\tau}{\sqrt{(\tau)}}. \tag{38}$$

By substituting equation (37) into equation (38), one obtains

$$\theta_0(\phi') = \frac{q''}{(\sqrt{\pi\rho CVK})}\left(\frac{4R^2}{3\phi'_\pi}\right)\int_0^{\phi'}\frac{d\tau}{\sqrt{\tau}\left\{2\cos\left[\frac{\cos^{-1}\left\{1 - 2\left[1 - 2\left(\frac{\phi' - \tau}{\phi'_\pi}\right)\right]^2\right\}}{3}\right] - 1\right\}}.$$

$$= \frac{q''}{(\sqrt{\pi\rho CVK})}\left(\frac{4R^2}{3\phi'_\pi}\right)I. \tag{39}$$

The integral, I, appearing in the above equation can be evaluated analytically by consecutive changes of variables in a proper manner. It can be shown that

$$I = 3\left(\sqrt{\frac{\phi'_\pi}{2}}\right)\int_{\sin\left[\frac{\sin^{-1}\left\{1 - 2\frac{\phi'}{\phi'_\pi}\right\}}{3}\right]}^{\frac{1}{2}}\frac{d\eta}{\sqrt{3\eta - 4\eta^3 - \left\{1 - 2\left(\frac{\phi'}{\phi'_\pi}\right)\right\}}}$$

$$= \frac{3}{2}\left(\sqrt{\frac{\phi'_\pi}{2}}\right)\frac{2}{3^{\frac{1}{4}}\left[\sqrt{\cos\left(\frac{\omega}{3} - \frac{\pi}{6}\right)}\right]}\left[F\left(\sin^{-1}K', \frac{\pi}{2}\right) - F(\sin^{-1}K', \xi)\right] \tag{40a}$$

where F designates the incomplete elliptic integral of the first kind and

$$\omega = \cos^{-1}\left\{2\left(\frac{\phi'}{\phi'_\pi}\right) - 1\right\} = \cos^{-1}\left\{\frac{\cos\theta(\cos^2\theta - 3)}{2}\right\} \tag{40b}$$

$$K' = \sqrt{\left[\cos\left(\frac{\omega}{3} + \frac{\pi}{6}\right)\Big/\cos\left(\frac{\omega}{3} - \frac{\pi}{6}\right)\right]} \tag{40c}$$

and

$$\xi = \sin^{-1}\left\{3^{-\frac{1}{4}}\sqrt{\left[\frac{\cos(\omega/3) - 0{\cdot}5}{\cos(\omega/3 + \pi/6)}\right]}\right\}. \tag{40d}$$

Combining equation (39) and equation (40d) thus yields:

$$\theta_0(\theta) = \frac{q''D^2}{2(\sqrt{2\pi})}\left(\sqrt{\frac{1}{KC\rho V}}\right)\frac{2}{3^{\frac{1}{4}}(\sqrt{\phi'_\pi})}\frac{1}{[\sqrt{\cos(\omega/3 - \pi/6)}]}\left[F\left(\sin^{-1}K', \frac{\pi}{2}\right) - F(\sin^{-1}K', \xi)\right] = \frac{q''D^2}{2(\sqrt{2\pi})}\left(\sqrt{\frac{1}{KC\rho V}}\right)\frac{F(\theta)}{\sqrt{(\phi'_\pi)}} \tag{41}$$

where

$$F(\theta) = \frac{2}{3^{\frac{1}{4}}\sqrt{\cos(\omega/3 - \pi/6)}}\left[F\left(\sin^{-1}K', \frac{\pi}{2}\right) - F(\sin^{-1}K', \xi)\right].$$

and, accordingly the local heat-transfer coefficient, $h_D(\theta)$, becomes

$$h_D(\theta) = \frac{q''}{\theta_0} = \left(\sqrt{\frac{\pi KC\rho V}{D}}\right)\frac{(\phi'_\pi/R^3)^{\frac{1}{4}}}{F(\theta)}. \tag{42}$$

The heat-transfer coefficient averaged over the surface of the sphere can therefore be obtained as:

$$\bar{h}_D = \frac{1}{4\pi R^2}\int_0^\pi\int_0^{2\pi} R^2\sin\theta\,[h_D(\theta)]\,d\beta\,d\theta = \frac{(\sqrt{\pi})}{2}\left(\sqrt{\frac{KC\rho V}{D}}\right)(\phi'_\pi/R^3)^{\frac{1}{4}}\int_0^\pi\frac{\sin\theta\,d\theta}{F(\theta)}$$

from which the following expression for $\overline{Nu}_D$ is obtained

$$\overline{Nu}_D = \frac{\bar{h}_D D}{K} = \frac{(\sqrt{\pi})}{2}Pe^{\frac{1}{2}}(\phi'_\pi/R^3)^{\frac{1}{4}}\int_0^\pi\frac{\sin\theta\,d\theta}{F(\theta)}. \tag{43}$$

The integral appearing in the above equation was evaluated numerically by using Simpson's rule. The average Nusselt number, $\overline{Nu}_D$, was finally obtained as:

$$\overline{Nu}_D = 1{\cdot}0035\,Pe^{\frac{1}{2}}(\phi'_\pi/R^3)^{\frac{1}{4}}. \tag{44}$$

To obtain the expression for $\overline{Nu}_t$, the temperature averaged over the surface of the sphere is first obtained.

$$\bar{\theta}_0 = \frac{1}{4\pi R^2}\int_0^\pi\int_0^{2\pi} R^2\sin\theta\,[\theta_0]\,d\beta\,d\theta = \frac{q''D^2}{4(\sqrt{2\pi\phi'_\pi})}\left(\frac{1}{KC\rho V}\right)^{\frac{1}{2}}\int_0^\pi F(\theta)\sin\theta\,d\theta.$$

Consequently,

$$h_t = \frac{q''}{\bar{\bar{\theta}}_0} = \frac{4(\sqrt{2\pi})(\sqrt{\phi'\pi})}{D^2}(KC\rho V)^{\frac{1}{2}} \Bigg/ \int_0^{\pi} F(\theta) \sin\theta \, d\theta$$

and

$$\overline{Nu}_t = \frac{h_t D}{K} = \frac{2(\sqrt{\pi})(\phi'_\pi/R^3)^{\frac{1}{2}} Pe^{\frac{1}{2}}}{\int_0^{\pi} F(\theta)\sin\theta \, d\theta}. \tag{45}$$

Once again, the integral appearing in the denominator of equation (45) was evaluated numerically, thus yielding

$$\overline{Nu}_t = 0{\cdot}9125\, Pe^{\frac{1}{2}} (\phi'_\pi/R^3)^{\frac{1}{2}}. \tag{46}$$

For flow past a single sphere, $\phi'_\pi/R^3 = 2{\cdot}0$ so that equations (44) and (46) reduce, respectively, to:

$$\overline{Nu}_D = 1{\cdot}4192\, Pe^{\frac{1}{2}} \tag{47}$$

$$\overline{Nu}_t = 1{\cdot}2904\, Pe^{\frac{1}{2}}. \tag{48}$$

DISCUSSION OF RESULTS AND CONCLUSIONS

In Figs. 2 and 3, the variation of the local Nusselt number as a function of θ, the angle measured from the forward stagnation point, is illustrated for the cases of uniform surface temperature and uniform surface heat flux, respectively. In both instances, the local Nusselt number, Nu_L, is seen to decrease from a maximum value at the forward stagnation point to a minimum at the rear stagnation point. This agrees qualitatively with the experimental observation of Hoe *et al.* [5] for cylinders. It can also be observed that Nu_L, in general, decreases as the spacing between the spheres is reduced. The broken line which is shown for comparison in each of these figures displays the behavior of Nu_L corresponding to the case of flow past a single circular cylinder [4]. The effect of spacing between cylinders cannot be shown because no heat-transfer analysis on this problem has been known. Comparison of the curve for a circular cylinder with that for a sphere ($P/D = \infty$) reveals that, for the case of uniform surface temperature, the variation of Nu_L is more gradual for the former than for the latter. The ratio of Nu_L at two angles, $(Nu_L)_{\theta=10°}/(Nu_L)_{\theta=120°}$, for instance, gives the value of ≈ 2 for the cylinder while it is about 2·8 for the sphere. It can also be perceived that, for a constant Péclet number, Nu_L is larger for the sphere than for the cylinder in the forward sections. This trend is reversed, however, at $\theta \approx 100°$. It should be cautioned that in calculating the average Nusselt number for the sphere, the local Nusselt number shown in the figures is not weighted evenly with respect to angle, θ. Unlike the case for a cylinder, the sphere does not extend uniformly in the z-direction. The weight of Nu_L, therefore, depends upon the local incremental area which can be generated by rotating the circle shown in the figures around the x–x' axis. The Nu_L at $\theta = 90°$, for instance, carries much larger weight than those for θ near 0° or 180°. For a circular cylinder, on the other hand, Nu_L at any angle θ weighs equally with respect to angle. By rigorous integration procedure, it was shown in the previous section [equation (32)] that for flow past a single sphere, the average Nusselt number, Nu is equal to $1{\cdot}128\, Pe^{\frac{1}{2}}$, in agreement with the result of Boussinesq's analysis [2]. For a circular cylinder, $Nu = 1{\cdot}015\, Pe^{\frac{1}{2}}$, as

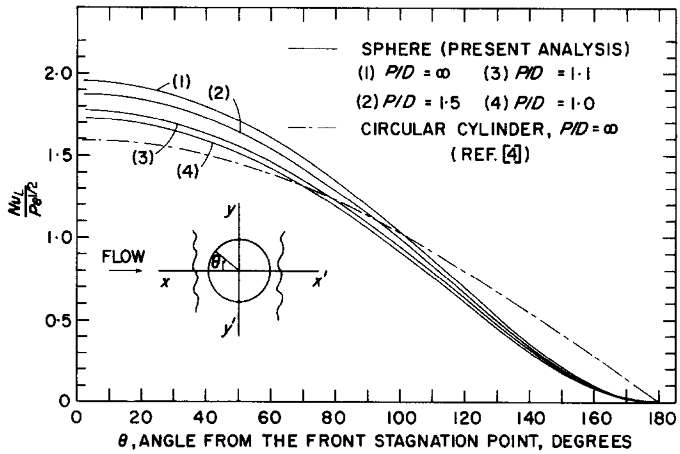

FIG. 2. Illustration of the variation of local Nusselt numbers for constant surface temperature case.

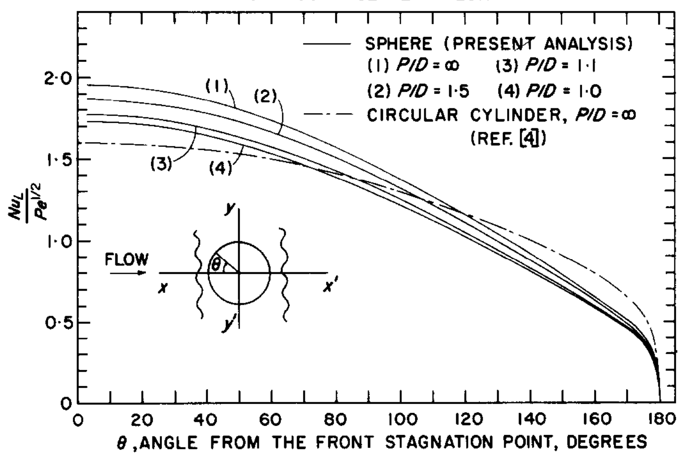

FIG. 3. Illustration of the variation of local Nusselt numbers for constant surface heat flux case.

reported by Grosh and Cess [4]. For the case of uniform surface heat flux, it can be noticed from Fig. 3 that the variation of Nu_L is relatively gradual both for the sphere and for the cylinder except for the part near the rear stagnation point, where abrupt lowering takes place. Once again, Nu_L is seen to be larger for a sphere in the front part. The reversal in this case, however, takes place at $\theta \approx 120°$.

Turning attention to the average Nusselt number, it can be concluded from equations (31, 44, 46), and the computational results tabulated in Table 2 that, for heat transfer to an inviscid fluid flowing along a row of spheres, the average Nusselt number generally decreases as the spacing between the spheres is reduced. Significant reduction in the average Nusselt number, however, does not occur until the pitch-to-diameter ratio, P/D, is reduced to about 2·0. As mentioned in the previous section, the surface velocity potential factor, ϕ'_π/R^3, is numerically equal to 2·0 for $P/D = \infty$. From Table 2, it can be noticed that its numerical value does not change significantly until P/D is lowered to about 2·0. For $P/D = 2{\cdot}0$, $\phi'_\pi/R^3 = 1{\cdot}9276$, and this implies roughly 2 per cent reduction in the average Nusselt number as compared with the case of flow past a single sphere. As P/D continues to diminish toward the limiting value of 1·0, the value of ϕ'_π/R^3 decreases rather rapidly and finally reaches the value of 1·5639 for a row of touching spheres. This amounts to ≈ 12 per cent reduction in the average Nusselt number. The relatively rapid decrease in the Nusselt number can be reasoned as follows. It should be remembered that the present heat-transfer analysis has been based upon the assumption of potential flow, in which the fluid stream lines are considered to be preserved everywhere in the flow region including the spaces between the spheres. If the spheres are comparatively far apart from each other, fluid can obviously dip into the space between the spheres. On the other hand, if the spacing between the spheres is small, the fluid tends to be excluded from the region near the x–x' axis (Fig 1), thus forming a nearly stagnant region. In fact, according to Michael's analysis [9], the region near the axis is essentially stagnant if the row of spheres are mutually touching. The formation of this fluid-dynamically stagnant region apparently will cause deterioration in the over-all heat-transfer effect, thus lowing the average Nusselt number. The present heat-transfer analysis predicts this tendency. Due to the lack of experimental information, however, verification of the soundness of this analysis has to rely upon future experimentation. In fact, if the spheres are actually touching each other, the postulation of negligible interference of thermal boundary layers between the adjacent spheres may no longer be valid. In this case, some deviation from the conclusion of this analysis may result. As a final remark, it should be mentioned that the validity of approximating the energy equation, equation (7), by equation (8), was examined in detail by Rigdon [11], for flow past a single isothermal sphere. According to Rigdon, the Boussinesq solution based on such an approximation falls within 1·5 per cent of the exact solution for a Péclet number of 500, and the deviation between the two solutions does not exceed 15 per cent when the Péclet number is lowered to 50. The Boussinesq solution for a single sphere, therefore, seems to be reasonably valid in the Péclet number range of 50–500. The validity of the present analysis is expected to be confined within the same Péclet number range, since the same approximation was made.

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Résumé—En supposant que l'écoulement est non visqueux et à potentiel on a analysé théoriquement le transport de chaleur vers des métaux liquides s'écoulant le long d'une rangée de sphères également espacées, dont le rapport du pas au diamètre varie depuis l'unité (sphères en contact) à l'infini (une seule sphère dans l'écoulement fluide). Les expressions explicites suivantes pour le nombre de Nusselt moyen ont été obtenues :

$$\text{Pour une température de surface uniforme: } \overline{Nu} = \left(\sqrt{\frac{2}{\pi}}\right) Pe^{\frac{1}{2}} (\phi'_\pi/R^3)^{\frac{1}{2}}$$

$$\text{Pour un flux de chaleur de surface uniforme: } \overline{Nu}_D = 1{,}0035\, Pe^{\frac{1}{2}} (\phi'_\pi/R^3)^{\frac{1}{2}}$$

On a trouvé que le facteur de potentiel de vitesse à la surface, ϕ'_π/R^3, qui apparaît dans les équations ci-dessus est fonction uniquement du rapport du pas au diamètre. On a exposé une méthode théorique d'évaluation des valeurs numériques de ϕ'_π/R^3, en utilisant le potentiel vectoriel obtenu récemment par Michael [9]. En liaison avec ceci, on a obtenu une équation décrivant le potentiel de vitesse le long d'une rangée de sphères.

Les valeurs numériques de ϕ'_π/R^3 ont été déterminées au moyen d'un calculateur numérique IBM. On a trouvé qu'elles varient de 2,0 pour l'écoulement le long d'une sphère unique à environ 1,5639 pour l'écoulement le long d'une rangée de sphères en contact.

Zusammenfassung—Unter der Voraussetzung der reibungslosen Potentialströmung wird eine theoretische Betrachtung über den Wärmeübergang zwischen flüssigen Metallen und einer Reihe von Kugeln, die gleichen Abstand von einander haben, angestellt. Das Verhältnis des Abstandes der Kugelmittelpunkte zum Kugeldurchmesser reicht von eins (sich berührende Kugeln) bis unendlich (einzelne Kugel in der Strömung). Folgende explizite Gleichungen für die mittlere Nusseltzahl werden angegeben:

$$\text{Für konstante Wandtemperatur } \overline{Nu} = \left(\sqrt{\frac{2}{\pi}}\right) Pe^{\frac{1}{2}} (\phi'_\pi/R^3)^{\frac{1}{2}}$$

$$\text{Für konstante Wärmestromdichte } \overline{Nu}_D = 1{,}0035\, Pe^{\frac{1}{2}} (\phi'_\pi/R^3)^{\frac{1}{2}}$$

Es zeigt sich, dass der Potentialfaktor der Oberflächengeschwindigkeit ϕ'_π/R^3, der in den obigen Gleichungen enthalten ist, nur eine Funktion des Verhältnisses der Kugelmittelpunktsabstände zum Kugeldurchmesser ist. Es wird eine theoretische Methode entwickelt, um die numerischen Werte von ϕ'_π/R^3 berechnen zu können. Dabei wird das Vektor-Potential benützt, das kürzlich von Michael [9] angegeben wurde. Im Zusammenhang damit wird eine Gleichung angegeben, die das Geschwindigkeitspotential entlang der Kugelreihe beschreibt. Zahlenwerte von ϕ'_π/R^3 wurden auf einer IBM Rechenanlage bestimmt. Die Zahlenwerte liegen zwischen 2,0 für die Strömung entlang einer einzelnen Kugel und 1,5639 für die Strömung entlang einer Reihe sich berührender Kugeln.

Аннотация—В допушении невязкого потока проведен теоретический анализ теплообмена ряда равно отстоящих друг от друга сфер, омываемых жидким металлом, относительный шаг которых изменяется от единицы (соприкасающиеся сферы) до бесконечности (единичная сфера в потоке жидкости). Получены следующие выражения для оценки среднего числа Нуссельта : для изотермической поверхности :

$$N\bar{u} = (\sqrt{2}/\pi)\, Pe^{\frac{1}{2}} (\varphi_\pi{}^6/R^3)^{\frac{1}{2}}$$

для равномерного теплового потока на поверхности :

$$Nu_D = 1{,}0035\, Pe^{\frac{1}{2}} (\phi_\pi\, R^3)^{\frac{1}{2}}$$

Найдено, что в указанных уравнениях коэффициент потенциала скорости на поверхности ϕ_π/R^3 является функцией только относительного шага. Для оценки численных значений ϕ_π/R^3 разработан теоретический метод, в котором использован недавно полученный в [9] потенциал вектора. В связи с этим выведено уравнение, описывающее потенциал скорости вдоль решетки сфер. Численные значения ϕ_π'/R^3 вычислялись на счетной машине. Найдено, что они изменяются от 2,0 для потока, омывающего единичную сферу, до $\approx$ 1,5639 для предельного случая касания сфер.